

Estimation Theory

UGC NET Statistics | Unit: Point Estimation | Concept + Formulas + Solved Examples

1. Definitions

ESTIMATOR AND ESTIMATE

An **estimator** $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$ is a statistic (function of the sample) used to approximate an unknown population parameter θ . A specific numerical value it takes for observed data is called an **estimate**.

2. Properties of a Good Estimator

UNBIASEDNESS

$\hat{\theta}$ is **unbiased** for θ if $E[\hat{\theta}] = \theta$ for all θ . Bias = $E[\hat{\theta}] - \theta$.

CONSISTENCY

$\hat{\theta}_n$ is **consistent** if $\hat{\theta}_n \xrightarrow{P} \theta$ as $n \rightarrow \infty$ (converges in probability). Sufficient condition: Bias $\rightarrow 0$ and $\text{Var}(\hat{\theta}_n) \rightarrow 0$.

EFFICIENCY

Among unbiased estimators, $\hat{\theta}_1$ is **more efficient** than $\hat{\theta}_2$ if $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$. The **most efficient** (minimum variance) unbiased estimator is called the **MVUE**.

SUFFICIENCY

$T(X_1, \dots, X_n)$ is **sufficient** for θ if the conditional distribution of the sample given T does not depend on θ . Tested via the **Factorization Theorem**:

$$f(x_1, \dots, x_n; \theta) = g(T(x); \theta) \cdot h(x_1, \dots, x_n)$$

3. Cramer-Rao Inequality

CRAMER-RAO LOWER BOUND (CRLB)

For an unbiased estimator $\hat{\theta}$ of θ , under regularity conditions:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n I(\theta)}, \quad I(\theta) = E \left[\left(\frac{\partial \log f(X; \theta)}{\partial \theta} \right)^2 \right]$$

where $I(\theta)$ is the **Fisher Information** per observation. An unbiased estimator achieving this bound with equality is called **efficient**.

4. Methods of Estimation

METHOD OF MOMENTS

Equate sample moments to population moments and solve for parameters:

$$\frac{1}{n} \sum_{i=1}^n X_i^k = E[X^k] \quad (k = 1, 2, \dots)$$

METHOD OF MAXIMUM LIKELIHOOD (MLE)

Given likelihood $L(\theta) = \prod_{i=1}^n f(x_i; \theta)$, the MLE $\hat{\theta}$ maximizes $L(\theta)$ (equivalently, maximizes $\log L(\theta)$):

$$\frac{\partial}{\partial \theta} \log L(\theta) = 0$$

INVARIANCE PROPERTY OF MLE

If $\hat{\theta}$ is the MLE of θ , then $g(\hat{\theta})$ is the MLE of $g(\theta)$ for any function g — a very useful shortcut (e.g. MLE of σ^2 gives MLE of σ as its square root).

5. Key Points to Remember

- Unbiasedness alone does not guarantee a good estimator — an unbiased estimator can have very high variance.
- MLE is **consistent** and **asymptotically efficient**, but may be **biased** for finite samples (e.g. MLE of σ^2 for normal population uses n , not $n - 1$, in the denominator, making it biased).
- Sample mean $\bar{X}$ is always unbiased for population mean μ , regardless of the underlying distribution.
- Sample variance with divisor $(n - 1)$ is unbiased for σ^2 ; with divisor n , it is biased (this is exactly why $(n - 1)$ appears in the standard formula).
- A sufficient statistic contains **all the information** in the sample relevant to θ — no other statistic based on the same sample can provide additional information about θ .
- Rao-Blackwell Theorem: given any unbiased estimator, conditioning on a sufficient statistic produces an estimator that is at least as good (lower or equal variance).

6. Worked Examples

EXAMPLE 1 — MLE OF POISSON PARAMETER

Find the MLE of λ for $X_1, \dots, X_n$ i.i.d. $\text{Poisson}(\lambda)$.

Solution: $L(\lambda) = \prod \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}.$

$$\log L(\lambda) = -n\lambda + \left(\sum x_i\right) \log \lambda - \sum \log(x_i!)$$

$$\frac{d}{d\lambda} \log L = -n + \frac{\sum x_i}{\lambda} = 0 \implies \hat{\lambda} = \bar{X}$$

So $\hat{\lambda}_{MLE} = \bar{X}$, which is also unbiased since $E[\bar{X}] = \lambda$.

EXAMPLE 2 — CHECKING UNBIASEDNESS

For $X_1, \dots, X_n$ i.i.d. with mean μ , is $T = \frac{X_1 + X_n}{2}$ unbiased for μ ?

Solution: $E[T] = \frac{E[X_1] + E[X_n]}{2} = \frac{\mu + \mu}{2} = \mu$. Yes, T is unbiased — but it is far less efficient than $\bar{X}$ since it ignores most of the data.

EXAMPLE 3 — METHOD OF MOMENTS FOR EXPONENTIAL

Find the method of moments estimator of λ for $\text{Exponential}(\lambda)$, where $E[X] = 1/\lambda$.

Solution: Equate: $\bar{X} = 1/\lambda \implies \hat{\lambda}_{MOM} = 1/\bar{X}$.

EXAMPLE 4 — APPLYING CRAMER-RAO BOUND

For $X_1, \dots, X_n$ i.i.d. $N(\mu, \sigma^2)$ with σ^2 known, find the CRLB for estimating μ .

Solution: Fisher information per observation: $I(\mu) = 1/\sigma^2$. So:

$$\text{Var}(\hat{\mu}) \geq \frac{1}{nI(\mu)} = \frac{\sigma^2}{n}$$

Since $\text{Var}(\bar{X}) = \sigma^2/n$ exactly, $\bar{X}$ **achieves** the CRLB, confirming it is the efficient (MVUE) estimator of μ .

7. Common Pitfalls (Exam Traps)

WATCH OUT

- MLE is not always unbiased — always check separately; the classic example is MLE of variance (biased, divides by n not $n - 1$).
- Unbiased $\neq$ good — always weigh variance/efficiency too, not just bias.
- The Cramer-Rao bound is a **lower bound** on variance of unbiased estimators — achieving it is sufficient (not necessary) for efficiency; not every efficient estimator problem has an estimator that attains the bound exactly.
- Consistency is an **asymptotic** (large-sample) property — a consistent estimator can still perform poorly for small n .

- Sufficiency does not imply unbiasedness or efficiency by itself — they are separate, complementary properties (though sufficiency plus Rao-Blackwell can improve any unbiased estimator).

8. Quick Revision Summary

Concept	Key Fact
Unbiased	$E[\hat{\theta}] = \theta$
Consistent	$\hat{\theta}_n \xrightarrow{P} \theta$
Efficient (MVUE)	Unbiased with minimum variance
Sufficiency test	Factorization Theorem
CRLB	$\text{Var}(\hat{\theta}) \geq 1/(nI(\theta))$
MLE property	Consistent, asymptotically efficient; may be biased
MLE invariance	$g(\hat{\theta}_{MLE})$ is MLE of $g(\theta)$
Rao-Blackwell	Conditioning on sufficient statistic improves (or preserves) unbiased estimators

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