

Probability — Basic Concepts & Bayes' Theorem

UGC NET Statistics | Unit: Probability Theory | Concept + Formulas + Solved Examples

1. Definitions

SAMPLE SPACE AND EVENT

The **sample space** S is the set of all possible outcomes of a random experiment. An **event** is any subset $A \subseteq S$. A and B are **mutually exclusive** if $A \cap B = \emptyset$.

AXIOMATIC DEFINITION (KOLMOGOROV)

A probability function P on S satisfies:

1. $P(A) \geq 0$ for every event A
2. $P(S) = 1$
3. For mutually exclusive $A_1, A_2, \dots$: $P(\bigcup_i A_i) = \sum_i P(A_i)$ (countable additivity)

2. Conditional Probability & Independence

CONDITIONAL PROBABILITY

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

MULTIPLICATION THEOREM

$$P(A \cap B) = P(A) P(B | A) = P(B) P(A | B)$$

For n events: $P(A_1 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap \dots \cap A_{n-1})$

INDEPENDENCE

A and B are **independent** if $P(A \cap B) = P(A)P(B)$, equivalently $P(A|B) = P(A)$.

INDEPENDENT $\neq$ MUTUALLY EXCLUSIVE

These are **very different** concepts. If A, B are mutually exclusive with $P(A), P(B) > 0$, then $P(A \cap B) = 0 \neq P(A)P(B)$, so mutually exclusive events with positive probability are **never independent**.

3. Bayes' Theorem

THEOREM — TOTAL PROBABILITY

If $B_1, \dots, B_n$ partition S (mutually exclusive, exhaustive), then for any event A :

$$P(A) = \sum_{i=1}^n P(B_i) P(A | B_i)$$

BAYES' THEOREM

$$P(B_i | A) = \frac{P(B_i) P(A | B_i)}{\sum_{j=1}^n P(B_j) P(A | B_j)}$$

Here $P(B_i)$ is the **prior** probability, and $P(B_i | A)$ is the **posterior** probability after observing A .

4. Key Points to Remember

- $P(\emptyset) = 0$, $P(A^c) = 1 - P(A)$, and $0 \leq P(A) \leq 1$ always.

- Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; for mutually exclusive events, $P(A \cup B) = P(A) + P(B)$.
- Pairwise independence of events does **not** imply mutual independence for 3 or more events.
- Bayes' theorem "reverses" conditioning — going from $P(A|B_i)$ (often known from a model) to $P(B_i|A)$ (what we want to infer).
- If $A \subseteq B$, then $P(A) \leq P(B)$ (monotonicity).
- Boole's inequality: $P(\bigcup_i A_i) \leq \sum_i P(A_i)$.

5. Worked Examples

EXAMPLE 1 — ADDITION RULE

If $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B) = 0.2$, find $P(A \cup B)$.

Solution: $P(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7$.

EXAMPLE 2 — CHECKING INDEPENDENCE

A fair die is rolled. $A = \{2, 4, 6\}$ (even), $B = \{1, 2, 3\}$ (at most 3). Are A, B independent?

Solution: $P(A) = 1/2$, $P(B) = 1/2$, $A \cap B = \{2\}$, so $P(A \cap B) = 1/6$. $P(A)P(B) = 1/4 \neq 1/6$. So A, B are **not independent**.

EXAMPLE 3 — BAYES' THEOREM (CLASSIC DISEASE TEST)

A disease affects 1% of a population. A test is 95% accurate for diseased people (true positive) and 90% accurate for healthy people (true negative). If a person tests positive, find the probability they actually have the disease.

Solution: Let D = has disease, T^+ = tests positive. $P(D) = 0.01$, $P(D^c) = 0.99$, $P(T^+|D) = 0.95$, $P(T^+|D^c) = 0.10$.

$$\begin{aligned}
 P(D|T^+) &= \frac{P(D)P(T^+|D)}{P(D)P(T^+|D) + P(D^c)P(T^+|D^c)} = \frac{0.01 \times 0.95}{0.01 \times 0.95 + 0.99 \times 0.10} \\
 &= \frac{0.0095}{0.0095 + 0.099} = \frac{0.0095}{0.1085} \approx 0.0876
 \end{aligned}$$

Only about **8.76%** — a strong illustration of why base rates matter (this surprises most students).

EXAMPLE 4 — TOTAL PROBABILITY

Three machines B_1, B_2, B_3 produce 30%, 45%, 25% of items respectively, with defect rates 2%, 3%, 4%. Find the overall probability an item is defective.

Solution:

$$P(\text{defective}) = 0.30(0.02) + 0.45(0.03) + 0.25(0.04) = 0.006 + 0.0135 + 0.01 = 0.0295$$

So about **2.95%** of items are defective overall.

6. Common Pitfalls (Exam Traps)

WATCH OUT

- Never assume mutually exclusive events are independent — mutually exclusive (with positive probability) events are always **dependent**.
- Bayes' theorem problems require the **partition** $B_1, \dots, B_n$ to be exhaustive and mutually exclusive — verify this before applying the formula.
- Low base-rate diseases/events give surprisingly low posterior probabilities even with "accurate" tests — always compute, don't guess intuitively.
- $P(A|B) \neq P(B|A)$ in general — confusing the two is one of the most common statistics exam errors (the "prosecutor's fallacy").

7. Quick Revision Summary

Concept	Key Formula
Addition rule	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Conditional probability	$P(A B) = P(A \cap B)/P(B)$

Independence	$P(A \cap B) = P(A)P(B)$
Total probability	$P(A) = \sum_i P(B_i)P(A B_i)$
Bayes' theorem	$P(B_i A) = \frac{P(B_i)P(A B_i)}{\sum_j P(B_j)P(A B_j)}$
Complement rule	$P(A^c) = 1 - P(A)$

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