

Random Variables & Probability Distributions

UGC NET Statistics | Unit: Probability Distributions | Concept + Formulas + Solved Examples

1. Definitions

RANDOM VARIABLE

A **random variable** X is a function $X : S \rightarrow \mathbb{R}$ from the sample space to the real numbers. X is **discrete** if it takes countably many values, and **continuous** if it takes values in an interval of $\mathbb{R}$.

PMF, PDF, CDF

For discrete X : **probability mass function** $p(x) = P(X = x)$, with $\sum_x p(x) = 1$. For continuous X : **probability density function** $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x) dx = 1$. For both: **cumulative distribution function** $F(x) = P(X \leq x)$.

2. Expectation, Variance, and Moments

EXPECTATION

$$E[X] = \sum_x x p(x) \text{ (discrete)}, \quad E[X] = \int_{-\infty}^{\infty} x f(x) dx \text{ (continuous)}$$

VARIANCE

$$\text{Var}(X) = E[X^2] - (E[X])^2 = E[(X - \mu)^2]$$

MOMENT GENERATING FUNCTION (MGF)

$M_X(t) = E[e^{tX}], \quad E[X^r] = M_X^{(r)}(0)$ (the r -th derivative at $t = 0$)

If X, Y independent: $M_{X+Y}(t) = M_X(t) M_Y(t)$.

3. Standard Discrete Distributions

Distribution	PMF	Mean	Variance
Bernoulli(p)	$p^x(1-p)^{1-x}, x = 0, 1$	p	$p(1-p)$
Binomial(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson(λ)	$\frac{e^{-\lambda} \lambda^x}{x!}$	λ	λ
Geometric(p)	$(1-p)^{x-1} p, x = 1, 2, \dots$	$1/p$	$(1-p)/p^2$

4. Standard Continuous Distributions

Distribution	PDF	Mean	Variance
Uniform(a, b)	$\frac{1}{b-a}, a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2
Exponential(λ)	$\lambda e^{-\lambda x}, x \geq 0$	$1/\lambda$	$1/\lambda^2$

POISSON AS LIMIT OF BINOMIAL

If $n \rightarrow \infty, p \rightarrow 0$ with $np = \lambda$ fixed, $\text{Binomial}(n, p) \rightarrow \text{Poisson}(\lambda)$. Used to approximate binomial probabilities when n is large and p is small.

5. Key Points to Remember

- $\text{Var}(X) \geq 0$ always; $\text{Var}(aX + b) = a^2 \text{Var}(X)$.

- $E[aX + b] = aE[X] + b$ (linearity of expectation), true even if X, Y are dependent.
- For independent X, Y : $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ — this fails if they are dependent (need to add $2\text{Cov}(X, Y)$ in general).
- Sum of independent Binomials with same p : $\text{Binomial}(n_1, p) + \text{Binomial}(n_2, p) \sim \text{Binomial}(n_1 + n_2, p)$.
- Sum of independent Poissons: $\text{Poisson}(\lambda_1) + \text{Poisson}(\lambda_2) \sim \text{Poisson}(\lambda_1 + \lambda_2)$.
- Sum of independent Normals: $N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2) \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.
- Memoryless property: only the Exponential (continuous) and Geometric (discrete) distributions have $P(X > s + t \mid X > s) = P(X > t)$.

6. Worked Examples

EXAMPLE 1 — MEAN AND VARIANCE FROM PMF

X takes values 0, 1, 2 with probabilities 0.2, 0.5, 0.3. Find $E[X]$ and $\text{Var}(X)$.

Solution: $E[X] = 0(0.2) + 1(0.5) + 2(0.3) = 0.5 + 0.6 = 1.1$. $E[X^2] = 0 + 1(0.5) + 4(0.3) = 0.5 + 1.2 = 1.7$. $\text{Var}(X) = 1.7 - (1.1)^2 = 1.7 - 1.21 = 0.49$.

EXAMPLE 2 — BINOMIAL PROBABILITY

A fair coin is tossed 10 times. Find $P(X = 6)$ where X = number of heads.

Solution: $X \sim \text{Binomial}(10, 0.5)$.

$$P(X = 6) = \binom{10}{6} (0.5)^6 (0.5)^4 = 210 \times (0.5)^{10} = \frac{210}{1024} \approx 0.205$$

EXAMPLE 3 — POISSON APPROXIMATION

A factory produces items with defect probability $p = 0.001$. In a batch of $n = 2000$, find the approximate probability of exactly 3 defectives.

Solution: $\lambda = np = 2000(0.001) = 2$. Using Poisson approximation:

$$P(X = 3) \approx \frac{e^{-2} 2^3}{3!} = \frac{e^{-2} \times 8}{6} \approx 0.180$$

EXAMPLE 4 — NORMAL DISTRIBUTION PROBABILITY

If $X \sim N(50, 100)$ (so $\sigma = 10$), find $P(X > 60)$.

Solution: $Z = \frac{X - 50}{10}$. $P(X > 60) = P\left(Z > \frac{60 - 50}{10}\right) = P(Z > 1)$. From standard normal tables, $P(Z > 1) \approx 0.1587$.

7. Common Pitfalls (Exam Traps)

WATCH OUT

- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ holds only for **independent** (or at least uncorrelated) X, Y — the general formula needs the covariance term.
- $E[X^2] \neq (E[X])^2$ in general — variance is exactly the gap between them.
- Poisson approximation to Binomial is valid only for **large n , small p** , with np moderate — don't apply it blindly for large p .
- Confusing PMF (discrete, gives probabilities directly) with PDF (continuous, gives probability **density**, not probability — $f(x)$ can exceed 1).
- Only Exponential and Geometric distributions are memoryless — do not assume this property for Normal, Binomial, etc.

8. Quick Revision Summary

Concept	Key Formula
Expectation	$E[X] = \sum xp(x)$ or $\int xf(x) dx$
Variance	$\text{Var}(X) = E[X^2] - (E[X])^2$
Binomial mean/var	$np, np(1 - p)$
Poisson mean/var	λ, λ

Normal mean/var	μ, σ^2
Exponential mean/var	$1/\lambda, 1/\lambda^2$
MGF sum rule	$M_{X+Y}(t) = M_X(t)M_Y(t)$ (independent)

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