

Compactness — CSIR NET Mathematical Sciences

Topology / Metric Spaces | Unit: Point-Set Topology | Concept + Formulas + Solved Examples

1. Definition

DEFINITION — OPEN COVER

Let X be a metric (or topological) space and $A \subseteq X$. A collection $\{U_\alpha\}_{\alpha \in I}$ of open sets is an **open cover** of A if $A \subseteq \bigcup_{\alpha \in I} U_\alpha$.

DEFINITION — COMPACT SET

A is **compact** if every open cover of A has a **finite subcover**: i.e. for every open cover $\{U_\alpha\}$ of A , there exist finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$A \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$$

DEFINITION — SEQUENTIAL COMPACTNESS

A is **sequentially compact** if every sequence in A has a subsequence converging to a point in A . In **metric spaces**, compactness $\iff$ sequential compactness (they coincide).

2. Fundamental Theorems

THEOREM — HEINE-BOREL THEOREM

A subset $A \subseteq \mathbb{R}^n$ is compact $\iff$ A is **closed and bounded**. (This equivalence is **specific to** $\mathbb{R}^n$ — it does not hold in general metric spaces.)

THEOREM — COMPACTNESS IS PRESERVED UNDER CONTINUOUS MAPS

If $f : X \rightarrow Y$ is continuous and $A \subseteq X$ is compact, then $f(A)$ is compact in Y .

COROLLARY — EXTREME VALUE THEOREM

A continuous real-valued function on a compact set attains its maximum and minimum values.

THEOREM — COMPACT SUBSETS ARE CLOSED AND BOUNDED (GENERAL METRIC SPACES)

In any metric space, every compact set is closed and bounded. However the **converse fails** in general (only holds in $\mathbb{R}^n$ by Heine-Borel).

3. Key Points to Remember

- A closed subset of a compact set is compact.
- A compact subset of a Hausdorff space is closed.
- Finite unions and arbitrary intersections of compact sets are compact.
- Continuous image of a compact set is compact — compactness is a **topological invariant**, preserved under homeomorphism.
- In infinite-dimensional normed spaces, the closed unit ball is **not compact** in general (unlike finite dimensions) — a key contrast tested in CSIR NET.
- A discrete infinite metric space is closed and bounded but **not compact** (open cover by singletons has no finite subcover) — showing Heine-Borel genuinely needs $\mathbb{R}^n$.
- Compactness generalizes finiteness: many properties of finite sets (attaining extrema, uniform continuity of continuous functions) extend to compact sets.

4. Important Results & Formulas

HEINE-BOREL (IN $\mathbb{R}^N$)

$A \text{ compact} \iff A \text{ closed and bounded}$

UNIFORM CONTINUITY ON COMPACT SETS

If $f : X \rightarrow Y$ is continuous and X is compact (metric space), then f is **uniformly continuous** on X .

TOTALLY BOUNDED + COMPLETE $\iff$ COMPACT

In a metric space, A is compact $\iff A$ is complete and **totally bounded** (can be covered by finitely many balls of any given radius $\varepsilon > 0$).

5. Worked Examples

EXAMPLE 1 — APPLYING HEINE-BOREL

Is $A = [0, 1] \cup [2, 3]$ compact in $\mathbb{R}$?

Solution: A is closed (finite union of closed sets) and bounded (contained in $[0, 3]$). By Heine-Borel, A is **compact**.

EXAMPLE 2 — BOUNDED BUT NOT CLOSED

Is $A = (0, 1)$ compact in $\mathbb{R}$?

Solution: A is bounded but not closed (does not contain its limit points 0, 1). By Heine-Borel, A is **not compact**. Explicitly, the open cover $\{(1/n, 1) : n \in \mathbb{N}\}$ has no finite subcover.

EXAMPLE 3 — CLOSED AND BOUNDED BUT NOT COMPACT (GENERAL METRIC SPACE)

Let X be an infinite set with the discrete metric $d(x, y) = 1$ for $x \neq y$. Is X compact?

Solution: X is bounded (all distances ≤ 1) and closed (in itself). But the open cover $\{\{x\} : x \in X\}$ (each singleton is open in the discrete metric) has **no finite subcover** since X is infinite. So X is **not compact** — showing Heine-Borel's equivalence is special to $\mathbb{R}^n$.

EXAMPLE 4 — CONTINUOUS IMAGE OF A COMPACT SET

Let $f(x) = x^2$ on $[-1, 2]$. Explain why f attains a maximum and minimum on this interval.

Solution: $[-1, 2]$ is closed and bounded, hence compact by Heine-Borel. Since f is continuous, $f([-1, 2])$ is compact, hence closed and bounded in $\mathbb{R}$ — so it attains its supremum and infimum (Extreme Value Theorem). Here $\min f = 0$ at $x = 0$, $\max f = 4$ at $x = 2$.

6. Common Pitfalls (Exam Traps)

WATCH OUT

- Heine-Borel (closed + bounded $\iff$ compact) is valid **only in $\mathbb{R}^n$** — in general metric spaces, closed and bounded does not imply compact (see Example 3).
- Compactness is **not** the same as closedness — a compact set is always closed (in Hausdorff spaces), but a closed set need not be compact (e.g. $\mathbb{R}$ itself is closed but unbounded, hence not compact).
- Open intervals are never compact in $\mathbb{R}$ — always check whether endpoints are included.
- In infinite-dimensional spaces, closed and bounded balls are typically **not** compact — a frequently tested contrast with finite-dimensional intuition.

7. Quick Revision Summary

Concept	Key Fact
Compact	Every open cover has a finite subcover
Heine-Borel ($\mathbb{R}^n$)	Compact $\iff$ closed and bounded
Sequential compactness	Equivalent to compactness in metric spaces
Continuous image	Compact $\rightarrow$ compact
Extreme Value Theorem	Continuous f on compact set attains max/min
Uniform continuity	Continuous on compact $\implies$ uniformly continuous

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