

Linear Transformations — CSIR NET

Mathematical Sciences

Linear Algebra | Unit: Linear Maps & Matrices | Concept + Formulas + Solved Examples

1. Definition

DEFINITION

Let V and W be vector spaces over a field F . A map $T : V \rightarrow W$ is called a **linear transformation** (or linear map) if for all $\alpha, \beta \in V$ and $a, b \in F$:

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

If $W = V$, T is called a **linear operator**. If $W = F$, T is called a **linear functional**.

2. Kernel and Image

DEFINITIONS

$$\ker(T) = \{v \in V : T(v) = 0\} \quad \text{Im}(T) = \{T(v) : v \in V\}$$

$\ker(T)$ is always a subspace of V ; $\text{Im}(T)$ is always a subspace of W .

THEOREM — RANK-NULLITY

If V is finite dimensional and $T : V \rightarrow W$ is linear:

$$\dim V = \dim(\ker T) + \dim(\text{Im } T)$$

i.e. $\text{nullity}(T) + \text{rank}(T) = \dim V$.

INJECTIVITY / SURJECTIVITY TEST

T is **injective (one-one)** $\iff \ker(T) = \{0\}$. T is **surjective (onto)** $\iff \operatorname{Im}(T) = W$. If $\dim V = \dim W$ (finite), then T injective $\iff T$ surjective $\iff T$ bijective (isomorphism).

3. Key Points to Remember

- Every linear transformation satisfies $T(0) = 0$.
- A linear transformation is completely determined by its action on a **basis** of V .
- If $\dim V = n$ and $\dim W = m$, then T can be represented by an $m \times n$ matrix once bases are fixed for V and W .
- Composition of linear transformations corresponds to **matrix multiplication**: if $T : V \rightarrow W$ and $S : W \rightarrow U$ are linear, then $S \circ T : V \rightarrow U$ is linear, and its matrix is the product of the matrices of S and T .
- An **isomorphism** is a bijective linear transformation; $V \cong W$ (isomorphic) iff $\dim V = \dim W$ (for finite-dimensional spaces over the same field).
- The set of all linear transformations $T : V \rightarrow W$, denoted $\operatorname{Hom}(V, W)$ or $L(V, W)$, is itself a vector space with $\dim L(V, W) = \dim V \cdot \dim W$.
- $\operatorname{Hom}(V, F)$ (linear functionals) is called the **dual space** V^* , with $\dim V^* = \dim V$.

4. Important Formulas

RANK-NULLITY

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

DIMENSION OF HOM SPACE

$$\dim L(V, W) = \dim V \times \dim W$$

CHANGE OF BASIS

If A is the matrix of T w.r.t. basis $\mathcal{B}_1$, and A' w.r.t. basis $\mathcal{B}_2$ (for an operator $T : V \rightarrow V$), with P the change-of-basis matrix:

$$A' = P^{-1}AP$$

5. Worked Examples

EXAMPLE 1 — PROJECTION MAP

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(x, y, z) = (x, y, 0)$ (projection onto the xy -plane).

Solution: T is linear (check: $T(a(x_1, y_1, z_1) + b(x_2, y_2, z_2)) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$ holds directly). $\ker(T) = \{(0, 0, z) : z \in \mathbb{R}\}$, so $\dim \ker T = 1$. $\text{Im}(T) = \{(x, y, 0)\}$, so $\dim \text{Im } T = 2$. Check: $1 + 2 = 3 = \dim \mathbb{R}^3$. ✓

EXAMPLE 2 — DIFFERENTIATION OPERATOR

Let $T : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$, $T(p) = p'$ (derivative).

Solution: T is linear by standard derivative rules. $\ker(T) = \{\text{constant polynomials}\}$, $\dim \ker T = 1$. Since every polynomial in P_2 arises as a derivative, T is surjective, so $\dim \text{Im } T = 3$. Check: $1 + 3 = 4 = \dim P_3(\mathbb{R})$. ✓

EXAMPLE 3 — TESTING NON-LINEARITY

Is $T : \mathbb{R}^2 \rightarrow \mathbb{R}$, $T(x, y) = xy$ linear?

Solution: $T(2(1, 1)) = T(2, 2) = 4$, but $2T(1, 1) = 2 \cdot 1 = 2$. Since $4 \neq 2$, T fails to satisfy scalar homogeneity, so T is **not linear**.

EXAMPLE 4 — MATRIX REPRESENTATION

Find the matrix of $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (2x + y, x - 3y)$ w.r.t. standard basis.

Solution: $T(1, 0) = (2, 1)$, $T(0, 1) = (1, -3)$. Writing these as columns:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

This A satisfies $T(v) = Av$ for all $v \in \mathbb{R}^2$.

6. Common Pitfalls (Exam Traps)

WATCH OUT

- Maps involving **products, squares, or constants added** (e.g. $T(x) = x^2$, $T(x, y) = (x + 1, y)$) are almost always non-linear — check $T(0) = 0$ first as a quick filter.
- Injective and surjective are **not equivalent** in general — they coincide only when domain and codomain have equal finite dimension.
- The matrix of a linear transformation depends on the **choice of basis** — always state which basis is being used.
- $\ker(T) = \{0\}$ means injective, not " $T(0) = 0$ " — every linear map sends $0 \rightarrow 0$; that alone says nothing about injectivity.

7. Quick Revision Summary

Concept	Key Fact
Definition	$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$
Rank-Nullity	$\text{rank}(T) + \text{nullity}(T) = \dim V$
Injective	$\ker(T) = \{0\}$
Surjective	$\text{Im}(T) = W$
Isomorphism condition	$\dim V = \dim W$ (finite-dim, same field)
$\dim L(V, W)$	$\dim V \times \dim W$
Change of basis	$A' = P^{-1}AP$

