

Sequences & Convergence — CSIR NET Mathematical Sciences

Real Analysis | Unit: Sequences and Series | Concept + Formulas + Solved Examples

1. Definition

DEFINITION — CONVERGENCE

A sequence (a_n) of real numbers is said to **converge** to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$n \geq N \implies |a_n - L| < \varepsilon$$

We write $a_n \rightarrow L$ or $\lim_{n \rightarrow \infty} a_n = L$. If no such L exists, the sequence **diverges**.

DEFINITION — CAUCHY SEQUENCE

(a_n) is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists N such that:

$$m, n \geq N \implies |a_n - a_m| < \varepsilon$$

2. Fundamental Theorems

THEOREM — COMPLETENESS OF $\mathbb{R}$ (CAUCHY CRITERION)

A sequence of real numbers converges $\iff$ it is a Cauchy sequence. (This equivalence is **special to complete spaces** like $\mathbb{R}$ and $\mathbb{C}$ — it fails in $\mathbb{Q}$.)

THEOREM — MONOTONE CONVERGENCE THEOREM

Every **bounded monotonic** sequence of real numbers converges. (Monotonic increasing + bounded above $\implies$ converges to its supremum; monotonic decreasing + bounded below

$\implies$ converges to its infimum.)

THEOREM — BOLZANO-WEIERSTRASS

Every bounded sequence of real numbers has a **convergent subsequence**.

KEY IMPLICATION CHAIN

Convergent $\implies$ Bounded $\implies$ has a convergent subsequence (Bolzano-Weierstrass).

Convergent $\iff$ Cauchy (in $\mathbb{R}$). None of these reverse in general — bounded does **not** imply convergent.

3. Key Points to Remember

- The limit of a convergent sequence is **unique**.
- Every convergent sequence is bounded, but the converse is false (e.g. $(-1)^n$).
- A sequence can be bounded and have a convergent subsequence yet still not converge itself.
- Algebra of limits: if $a_n \rightarrow L$ and $b_n \rightarrow M$, then $a_n + b_n \rightarrow L + M$, $a_n b_n \rightarrow LM$, and $a_n/b_n \rightarrow L/M$ (provided $M \neq 0$).
- Sandwich (Squeeze) Theorem: if $a_n \leq b_n \leq c_n$ eventually, and $a_n, c_n \rightarrow L$, then $b_n \rightarrow L$.
- A sequence with two subsequences converging to **different** limits does not converge.
- Cauchy sequences are always bounded.

4. Important Formulas / Standard Limits

STANDARD LIMITS

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \ (p > 0), \quad \lim_{n \rightarrow \infty} n^{1/n} = 1$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x, \quad \lim_{n \rightarrow \infty} x^{1/n} = 1 \ (x > 0)$$

$$\lim_{n \rightarrow \infty} r^n = 0 \text{ if } |r| < 1, \quad \text{diverges if } |r| > 1$$

SERIES CONVERGENCE TESTS

Ratio Test: for $\sum a_n$, let $L = \lim \left| \frac{a_{n+1}}{a_n} \right|$. Converges (absolutely) if $L < 1$; diverges if $L > 1$; inconclusive if $L = 1$.

Root Test: let $L = \lim |a_n|^{1/n}$. Same conclusion rules as ratio test.

p -series: $\sum \frac{1}{n^p}$ converges $\iff p > 1$.

5. Worked Examples

EXAMPLE 1 — BASIC LIMIT VIA DEFINITION

Show $a_n = \frac{n+1}{n} \rightarrow 1$.

Solution: $|a_n - 1| = \left| \frac{n+1}{n} - 1 \right| = \frac{1}{n}$. Given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$. Then for $n \geq N$, $\frac{1}{n} < \varepsilon$. Hence $a_n \rightarrow 1$.

EXAMPLE 2 — BOUNDED BUT DIVERGENT

Show $a_n = (-1)^n$ does not converge.

Solution: The subsequence $a_{2n} = 1 \rightarrow 1$, while $a_{2n+1} = -1 \rightarrow -1$. Since two subsequences converge to different limits, (a_n) **diverges**, even though it is bounded ($|a_n| \leq 1$ for all n).

EXAMPLE 3 — MONOTONE CONVERGENCE THEOREM APPLICATION

Show $a_n = \left(1 + \frac{1}{n}\right)^n$ converges (this limit defines e).

Solution: One shows (via binomial expansion) that (a_n) is monotonically increasing and bounded above by 3. By the Monotone Convergence Theorem, (a_n) converges. Its limit is

defined to be $e \approx 2.718$.

EXAMPLE 4 — RATIO TEST ON A SERIES

Test $\sum_{n=1}^{\infty} \frac{n}{2^n}$ for convergence.

Solution:

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)/2^{n+1}}{n/2^n} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2} < 1$$

Since $L < 1$, the series **converges** by the Ratio Test.

6. Common Pitfalls (Exam Traps)

WATCH OUT

- Bounded $\not\Rightarrow$ convergent — always look for oscillation before concluding convergence from boundedness alone.
- The Cauchy $\iff$ convergent equivalence holds in $\mathbb{R}$ (complete space) but **fails** in incomplete spaces like $\mathbb{Q}$ — a classic CSIR NET conceptual trap.
- Ratio/root test giving $L = 1$ is **inconclusive** — do not default to "diverges"; use another test (e.g. p -series comparison, integral test).
- Absolute convergence $\implies$ convergence, but convergence does not imply absolute convergence (e.g. alternating harmonic series $\sum (-1)^n/n$).

7. Quick Revision Summary

Concept	Key Fact
Convergence	$\forall \varepsilon > 0, \exists N, n \geq N \implies a_n - L < \varepsilon$
Cauchy $\iff$ Convergent	True in $\mathbb{R}$ (completeness); false in $\mathbb{Q}$

Monotone Convergence	Bounded + monotonic $\implies$ convergent
Bolzano-Weierstrass	Bounded $\implies$ has convergent subsequence
Ratio Test	$L < 1$ converge, $L > 1$ diverge, $L = 1$ inconclusive
p -series	Converges iff $p > 1$

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