

# Analytic Functions & Cauchy-Riemann Equations — CSIR NET Mathematical Sciences

Complex Analysis | Unit: Analyticity & Differentiability | Concept + Formulas + Solved Examples

## 1. Definition

### DEFINITION — COMPLEX DIFFERENTIABILITY

A function  $f : D \rightarrow \mathbb{C}$  ( $D$  open in  $\mathbb{C}$ ) is **differentiable** at  $z_0 \in D$  if:

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, where  $h \in \mathbb{C}$  approaches 0 from **any direction**.

### DEFINITION — ANALYTIC (HOLOMORPHIC) FUNCTION

$f$  is **analytic** (or holomorphic) at  $z_0$  if  $f$  is differentiable in some open neighbourhood of  $z_0$  — not just at the single point.  $f$  is **entire** if it is analytic on all of  $\mathbb{C}$ .

## 2. Cauchy-Riemann Equations

### THEOREM — NECESSARY CONDITION

Write  $f(z) = u(x, y) + iv(x, y)$  where  $z = x + iy$ . If  $f$  is differentiable at  $z_0 = (x_0, y_0)$ , then the partial derivatives of  $u, v$  exist at  $(x_0, y_0)$  and satisfy the **Cauchy-Riemann (CR) equations**:

$$u_x = v_y, \quad u_y = -v_x$$

### THEOREM — SUFFICIENT CONDITION

If  $u, v$  have continuous first-order partial derivatives in a neighbourhood of  $(x_0, y_0)$  and satisfy the CR equations there, then  $f = u + iv$  is **differentiable** (in fact analytic) at  $z_0$ .

### CR EQUATIONS ALONE ARE NOT SUFFICIENT

Satisfying the CR equations at a point, **without continuity** of the partial derivatives, does not guarantee differentiability — a classic CSIR NET trap (see Example 3 below).

## 3. Key Points to Remember

- If  $f$  is analytic,  $u$  and  $v$  are both **harmonic**:  $u_{xx} + u_{yy} = 0$  and  $v_{xx} + v_{yy} = 0$ .
- $v$  is called the **harmonic conjugate** of  $u$  (and vice versa, up to a constant).
- $f'(z) = u_x + iv_x = v_y - iu_y$  (both formulas equivalent by CR equations).
- A non-constant analytic function on a domain is an **open map** (Open Mapping Theorem) and cannot attain a local max of  $|f|$  inside the domain (Maximum Modulus Principle).
- If  $f$  is analytic and  $f'(z) = 0$  everywhere in a domain, then  $f$  is constant.
- $f(z) = \bar{z}$ ,  $f(z) = \operatorname{Re}(z)$ ,  $f(z) = |z|$  are all **nowhere analytic** (though  $\bar{z}$  is real-differentiable as a map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ).
- Polynomial functions in  $z$  are entire; rational functions are analytic except at poles (zeros of the denominator).

## 4. Important Formulas

### CR EQUATIONS (CARTESIAN)

$$u_x = v_y, \quad u_y = -v_x$$

### CR EQUATIONS (POLAR FORM, $Z = RE^{i\theta}$ )

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

### DERIVATIVE VIA CR

$$f'(z) = u_x + iv_x$$

### LAPLACE EQUATION (HARMONICITY)

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

## 5. Worked Examples

### EXAMPLE 1 — VERIFY ANALYTICITY OF $F(Z) = Z^2$

Show  $f(z) = z^2$  is entire.

**Solution:**  $z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ , so  $u = x^2 - y^2$ ,  $v = 2xy$ .  $u_x = 2x$ ,  $v_y = 2x$  ✓;  $u_y = -2y$ ,  $-v_x = -2y$  ✓. CR equations hold everywhere, and partials are continuous everywhere, so  $f$  is entire, with  $f'(z) = u_x + iv_x = 2x + 2iy = 2z$ .

### EXAMPLE 2 — SHOWING $F(Z) = \bar{Z}$ IS NOWHERE ANALYTIC

Show  $f(z) = \bar{z} = x - iy$  is not differentiable anywhere.

**Solution:**  $u = x$ ,  $v = -y$ .  $u_x = 1$ ,  $v_y = -1$ . Since  $u_x \neq v_y$  ( $1 \neq -1$ ) at every point, the CR equations **fail everywhere**, so  $f$  is nowhere differentiable (hence nowhere analytic).

### EXAMPLE 3 — CR HOLDS BUT NOT DIFFERENTIABLE (CLASSIC TRAP)

Consider  $f(z) = \sqrt{|xy|}$  (i.e.  $u(x, y) = \sqrt{|xy|}$ ,  $v = 0$ ). It can be shown  $u_x = u_y = 0$  at the origin, so CR equations hold at  $z = 0$ . However, along  $z = t(1 + i)$ , the difference quotient does not tend to a single limit as  $t \rightarrow 0$ .

**Conclusion:**  $f$  is **not differentiable** at 0 despite satisfying CR there — because the partial derivatives of  $u$  are **not continuous** at the origin. This illustrates why the sufficient condition

requires continuity of partials, not just the equations holding pointwise.

#### EXAMPLE 4 — FINDING A HARMONIC CONJUGATE

Given  $u(x, y) = x^2 - y^2$ , find  $v$  such that  $f = u + iv$  is analytic.

**Solution:** From CR:  $v_y = u_x = 2x \implies v = 2xy + g(x)$ . Also  $v_x = -u_y = 2y \implies 2y + g'(x) = 2y \implies g'(x) = 0 \implies g(x) = c$ .

So  $v(x, y) = 2xy + c$ , and  $f(z) = (x^2 - y^2) + i(2xy + c) = z^2 + ic$ , an entire function.

## 6. Common Pitfalls (Exam Traps)

### WATCH OUT

- CR equations holding at a point is **necessary but not sufficient** for differentiability there — always check continuity of the partial derivatives too (see Example 3).
- Differentiable at a single isolated point does **not** mean analytic — analyticity requires differentiability throughout a neighbourhood.
- Real-differentiability of  $u, v$  as functions of  $(x, y)$  is not the same as complex-differentiability of  $f$  — the CR equations are the bridge between the two.
- Do not confuse harmonic (satisfies Laplace's equation) with analytic — every component of an analytic function is harmonic, but an arbitrary harmonic function alone is not "analytic" (only  $u + iv$  together, with  $v$  a genuine harmonic conjugate, is).

## 7. Quick Revision Summary

### Concept

### Key Fact

CR equations

$$u_x = v_y, \quad u_y = -v_x$$

Necessary condition

Differentiable  $\implies$  CR equations hold

|                      |                                                         |
|----------------------|---------------------------------------------------------|
| Sufficient condition | CR + continuous partials $\implies$ differentiable      |
| Derivative formula   | $f'(z) = u_x + iv_x$                                    |
| Analytic $\implies$  | $u, v$ harmonic; $f$ is an open map                     |
| $f(z) = \bar{z}$     | Nowhere differentiable                                  |
| Polar CR             | $u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta$ |

CSIR NET

Complex Analysis

Analytic Functions

Cauchy-Riemann