

# Diagonalization — CSIR NET Mathematical Sciences

Linear Algebra | Unit: Eigenvalues, Eigenvectors & Diagonalization | Concept + Formulas + Solved Examples

## 1. Definition

### DEFINITION

A square matrix  $A \in M_n(F)$  is said to be **diagonalizable** if there exists an invertible matrix  $P$  and a diagonal matrix  $D$  such that:

$$P^{-1}AP = D$$

Equivalently, a linear operator  $T : V \rightarrow V$  is diagonalizable if  $V$  has a basis consisting entirely of eigenvectors of  $T$  (called an **eigenbasis**).

Diagonalization expresses  $A$  as **similar** to a diagonal matrix  $D$  whose diagonal entries are precisely the eigenvalues of  $A$ , each repeated according to how many independent eigenvectors it contributes.

## 2. Diagonalizability Criterion

### THEOREM — MAIN CRITERION

An  $n \times n$  matrix  $A$  is diagonalizable over  $F$  if and only if  $A$  has  $n$  linearly independent eigenvectors in  $F^n$ , i.e.

$$\sum_i \dim E_{\lambda_i} = n$$

where  $E_{\lambda_i} = \{v : Av = \lambda_i v\}$  is the eigenspace for eigenvalue  $\lambda_i$ .

### THEOREM — ALGEBRAIC VS GEOMETRIC MULTIPLICITY

For each eigenvalue  $\lambda_i$ , let  $a_i$  = algebraic multiplicity (multiplicity of  $\lambda_i$  as a root of the characteristic polynomial) and  $g_i$  = geometric multiplicity ( $\dim E_{\lambda_i}$ ). Always:

$$1 \leq g_i \leq a_i$$

$A$  is diagonalizable  $\iff g_i = a_i$  for every eigenvalue  $\lambda_i$ .

#### SUFFICIENT (NOT NECESSARY) CONDITION

If  $A \in M_n(F)$  has  $n$  **distinct** eigenvalues, then  $A$  is automatically diagonalizable (since distinct eigenvalues always give linearly independent eigenvectors). The converse is **false** — repeated eigenvalues can still be diagonalizable if  $g_i = a_i$ .

### 3. Key Points to Remember

- Diagonalization requires eigenvectors to lie in  $F^n$  — a matrix diagonalizable over  $\mathbb{C}$  may **not** be diagonalizable over  $\mathbb{R}$  (e.g. rotation matrices).
- If  $P^{-1}AP = D$ , the columns of  $P$  are eigenvectors of  $A$ , and the diagonal entries of  $D$  are the corresponding eigenvalues, in the **same order**.
- Diagonalizability depends only on eigenspace dimensions, not on whether  $A$  is invertible — a diagonalizable matrix can have 0 as an eigenvalue.
- Every **real symmetric matrix** is diagonalizable, and in fact orthogonally diagonalizable (Spectral Theorem).
- Every **Hermitian matrix** (complex analogue,  $A^* = A$ ) is diagonalizable via a unitary matrix.
- A **defective** matrix is one that is *not* diagonalizable — it has at least one eigenvalue with  $g_i < a_i$ .
- Nilpotent matrices (other than the zero matrix) are **never diagonalizable**, since all eigenvalues are 0 but the matrix itself is non-zero.
- Diagonalization is unique only up to the **order** of eigenvalues on the diagonal and choice of eigenvector basis —  $P$  and  $D$  are not unique.

### 4. Important Formulas & Results

#### 4.1 Powers of a Diagonalizable Matrix

### FORMULA

If  $A = PDP^{-1}$ , then for any positive integer  $k$ :

$$A^k = PD^kP^{-1}$$

where  $D^k$  is obtained simply by raising each diagonal entry of  $D$  to the power  $k$ . This is the single biggest reason diagonalization is used in CSIR NET problems — computing  $A^{100}$  directly is hard, but  $D^{100}$  is trivial.

## 4.2 Determinant and Trace

### FORMULAS

$$\det(A) = \prod_{i=1}^n \lambda_i \quad \text{tr}(A) = \sum_{i=1}^n \lambda_i$$

(True for any square matrix, not just diagonalizable ones — eigenvalues counted with algebraic multiplicity.)

## 4.3 Diagonalizing Symmetric / Hermitian Matrices

### SPECTRAL THEOREM

If  $A$  is real symmetric, there exists an **orthogonal** matrix  $Q$  ( $Q^T Q = I$ ) such that:

$$Q^T A Q = D$$

with  $D$  diagonal and real. Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are automatically **orthogonal**.

## 4.4 Similarity and Invariants

### FORMULA

If  $A$  and  $B$  are similar ( $B = P^{-1}AP$ ), they share:

$$\det(A) = \det(B), \quad \text{tr}(A) = \text{tr}(B), \quad \text{rank}(A) = \text{rank}(B),$$

characteristic polynomial of  $A$  = characteristic polynomial of  $B$

## 5. Worked Examples

### EXAMPLE 1 — DIAGONALIZE A $2 \times 2$ MATRIX

Diagonalize  $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ .

**Solution:** Characteristic polynomial:

$$\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0 \implies \lambda = 5, 2$$

Since eigenvalues are distinct,  $A$  is diagonalizable. For  $\lambda = 5$ :  $(A - 5I)v = 0$  gives eigenvector  $v_1 = (1, 1)^T$ . For  $\lambda = 2$ : eigenvector  $v_2 = (1, -2)^T$ .

So  $P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ ,  $D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ , and  $P^{-1}AP = D$ .

### EXAMPLE 2 — A NON-DIAGONALIZABLE (DEFECTIVE) MATRIX

Check whether  $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$  is diagonalizable.

**Solution:** Characteristic polynomial:  $(2 - \lambda)^2 = 0 \implies \lambda = 2$  with algebraic multiplicity  $a_1 = 2$ .

Eigenspace:  $(A - 2I)v = 0 \implies \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v = 0 \implies v = (t, 0)^T$ . So  $\dim E_2 = 1$ , meaning geometric multiplicity  $g_1 = 1 < a_1 = 2$ .

Since  $g_1 \neq a_1$ ,  $A$  is **not diagonalizable** (it is defective).

### EXAMPLE 3 — USING DIAGONALIZATION TO COMPUTE $A^N$

Using Example 1's result, compute  $A^{10}$  for  $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ .

**Solution:**  $A^{10} = PD^{10}P^{-1}$  where  $D^{10} = \begin{pmatrix} 5^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}$ .

Substituting  $P$  and  $P^{-1} = \frac{1}{-3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}$  and multiplying out gives  $A^{10}$  directly — far easier than repeated matrix multiplication.

### EXAMPLE 4 — SYMMETRIC MATRIX (SPECTRAL THEOREM)

Show  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  is orthogonally diagonalizable.

**Solution:**  $A$  is symmetric, so the Spectral Theorem guarantees diagonalizability. Eigenvalues:  $(2 - \lambda)^2 - 1 = 0 \implies \lambda = 3, 1$ .

Eigenvectors: for  $\lambda = 3$ ,  $v_1 = (1, 1)^T$ ; for  $\lambda = 1$ ,  $v_2 = (1, -1)^T$  — already orthogonal, as guaranteed by the theorem. Normalizing gives orthogonal matrix  $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  with  $Q^T A Q = \text{diag}(3, 1)$ .

## 6. Common Pitfalls (Exam Traps)

### WATCH OUT

- Repeated eigenvalues do **not** automatically mean non-diagonalizable — always check  $g_i$  vs  $a_i$  before concluding.
- Distinct eigenvalues  $\implies$  diagonalizable, but diagonalizable  $\not\implies$  distinct eigenvalues (identity matrix is diagonalizable with a single repeated eigenvalue).
- Diagonalizability is **field-dependent** — check whether the question asks over  $\mathbb{R}$  or  $\mathbb{C}$ . Real eigenvalues may not exist over  $\mathbb{R}$  even though they exist over  $\mathbb{C}$ .
- The order of eigenvectors in  $P$  must exactly match the order of eigenvalues in  $D$ .
- A triangular matrix with all diagonal entries equal (and off-diagonal non-zero) is typically **not** diagonalizable — classic defective matrix pattern (see Example 2).
- Orthogonal diagonalization ( $Q^T A Q = D$ ) only works for **symmetric** (or Hermitian) matrices — not every diagonalizable matrix is orthogonally diagonalizable.

## 7. Quick Revision Summary

### Concept

### Key Formula / Condition

Diagonalization

$P^{-1} A P = D$ ,  $P$  invertible,  $D$  diagonal

|                      |                                                                   |
|----------------------|-------------------------------------------------------------------|
| Main criterion       | $n$ linearly independent eigenvectors, i.e. $\sum g_i = n$        |
| Multiplicities       | $1 \leq g_i \leq a_i$ ; diagonalizable $\iff g_i = a_i \forall i$ |
| Sufficient condition | $n$ distinct eigenvalues $\implies$ diagonalizable                |
| Powers               | $A^k = P D^k P^{-1}$                                              |
| Determinant / Trace  | $\det A = \prod \lambda_i, \operatorname{tr}(A) = \sum \lambda_i$ |
| Symmetric matrices   | Always diagonalizable; orthogonally via Spectral Theorem          |
| Nilpotent matrices   | Never diagonalizable (except zero matrix)                         |

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