

Vector Space — CSIR NET Mathematical Sciences

Linear Algebra | Unit: Vector Spaces | Concept + Formulas + Solved Examples

1. Definition

DEFINITION

Let F be a field. A non-empty set V together with two operations — **vector addition** $(+ : V \times V \rightarrow V)$ and **scalar multiplication** $(\cdot : F \times V \rightarrow V)$ — is called a **vector space** (or linear space) over F if the following axioms hold for all $\alpha, \beta, \gamma \in V$ and $a, b \in F$.

2. The Ten Axioms

2.1 Axioms for Vector Addition ($V, +$) is an Abelian Group

ADDITION AXIOMS

1. **Closure:** $\alpha + \beta \in V$
2. **Associativity:** $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
3. **Commutativity:** $\alpha + \beta = \beta + \alpha$
4. **Existence of zero vector:** $\exists 0 \in V$ such that $\alpha + 0 = \alpha$ for all $\alpha \in V$
5. **Existence of additive inverse:** $\forall \alpha \in V, \exists (-\alpha) \in V$ such that $\alpha + (-\alpha) = 0$

2.2 Axioms for Scalar Multiplication

SCALAR MULTIPLICATION AXIOMS

6. **Closure:** $a\alpha \in V$ for $a \in F, \alpha \in V$
7. **Distributivity over vector addition:** $a(\alpha + \beta) = a\alpha + a\beta$

- 8. **Distributivity over scalar addition:** $(a + b)\alpha = a\alpha + b\alpha$
- 9. **Associativity of scalars:** $a(b\alpha) = (ab)\alpha$
- 10. **Identity element:** $1 \cdot \alpha = \alpha$, where 1 is the multiplicative identity of F

QUICK MEMORY AID

Think of V as an **abelian group under addition** (axioms 1–5), on which the field F **acts compatibly** via scalar multiplication (axioms 6–10). The elements of V are called **vectors**, and the elements of F are called **scalars**.

3. Key Points to Remember

- The zero vector $0 \in V$ is **unique**, and the additive inverse $-\alpha$ of each α is **unique**.
- For all $\alpha \in V$: $0 \cdot \alpha = 0$ (scalar zero times any vector gives the zero vector).
- For all $a \in F$: $a \cdot 0 = 0$ (any scalar times the zero vector gives the zero vector).
- $(-1)\alpha = -\alpha$ for every $\alpha \in V$.
- $a\alpha = 0 \implies a = 0$ or $\alpha = 0$ (no zero divisors between scalars and vectors).
- A vector space is always defined **relative to a field** — the same set can be a vector space over one field but not another (e.g. $\mathbb{C}$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$ itself, with different dimensions).
- If $F = \mathbb{R}$, V is called a **real vector space**; if $F = \mathbb{C}$, it is a **complex vector space**.
- Every field is a vector space over itself (and over any subfield of itself).

4. Standard Examples

Vector Space	Description	Dimension (over F)
F^n	n -tuples $(a_1, \dots, a_n)$, $a_i \in F$	n
$M_{m \times n}(F)$	$m \times n$ matrices over F	mn

$P_n(F)$	Polynomials of degree $\leq n$ over F	$n + 1$
$P(F)$	All polynomials over F (no degree bound)	∞ (countable basis)
$C[a, b]$	Continuous real functions on $[a, b]$	∞
$\mathbb{C}$ over $\mathbb{R}$	Complex numbers as an $\mathbb{R}$ -vector space	2
$\{0\}$	The zero vector space (trivial)	0

5. Important Formulas & Results

5.1 Basic Identities

FORMULAS

$$\begin{aligned}
 0 \cdot \alpha &= 0, & a \cdot 0 &= 0, & (-1)\alpha &= -\alpha \\
 a\alpha &= 0 \iff a = 0 \text{ or } \alpha = 0
 \end{aligned}$$

5.2 Dimension of F^n over a Subfield

FORMULA

If F is a field extension of K with $[F : K] = m$ (i.e. F has dimension m as a vector space over K), then:

$$\dim_K F^n = mn$$

Example: $\dim_{\mathbb{R}} \mathbb{C}^n = 2n$, since $[\mathbb{C} : \mathbb{R}] = 2$.

5.3 Dimension of Matrix and Polynomial Spaces

FORMULAS

$$\dim M_{m \times n}(F) = mn \qquad \dim P_n(F) = n + 1$$

6. Worked Examples

EXAMPLE 1 — VERIFYING A VECTOR SPACE

Check whether $V = \mathbb{R}^2$ with addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and scalar multiplication $a(x, y) = (ax, ay)$ is a vector space over $\mathbb{R}$.

Solution: This is the standard component-wise addition and scaling on $\mathbb{R}^2$. All 10 axioms hold directly (associativity/commutativity of real addition, zero vector $(0, 0)$, inverse $(-x, -y)$, and distributive/associative laws for scalars follow from properties of $\mathbb{R}$). Hence V is a vector space over $\mathbb{R}$, with $\dim V = 2$.

EXAMPLE 2 — A NON-EXAMPLE (EXAM FAVOURITE)

Check whether $V = \mathbb{R}^2$ with the *unusual* scalar multiplication $a(x, y) = (ax, 0)$ (keeping ordinary addition) is a vector space over $\mathbb{R}$.

Solution: Test axiom 10 (identity): $1 \cdot (x, y) = (x, 0) \neq (x, y)$ unless $y = 0$.

Since $1 \cdot \alpha \neq \alpha$ in general, axiom 10 **fails**. Hence V with this operation is **not** a vector space.

EXAMPLE 3 — VECTOR SPACE OF FUNCTIONS

Let $V = C[0, 1]$, the set of all continuous real-valued functions on $[0, 1]$, with $(f + g)(x) = f(x) + g(x)$ and $(af)(x) = a f(x)$.

Solution: The zero vector is the zero function $f(x) = 0$. Sum and scalar multiples of continuous functions are continuous (closure holds), and all algebraic axioms follow from the corresponding properties of real numbers pointwise. Hence $C[0, 1]$ is a vector space over $\mathbb{R}$, and it is **infinite-dimensional**.

EXAMPLE 4 — $\mathbb{C}$ AS A VECTOR SPACE OVER $\mathbb{R}$

Show $\dim_{\mathbb{R}} \mathbb{C} = 2$.

Solution: Every $z \in \mathbb{C}$ can be written uniquely as $z = a \cdot 1 + b \cdot i$ with $a, b \in \mathbb{R}$. So $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$, giving $\dim_{\mathbb{R}} \mathbb{C} = 2$. (Note: $\dim_{\mathbb{C}} \mathbb{C} = 1$ — dimension always depends on the base field.)

7. Common Pitfalls (Exam Traps)

WATCH OUT

- Dimension is **relative to the base field** — always check "vector space over which field?" before quoting a dimension.
- Redefined or "twisted" operations (like Example 2) often break exactly **one** axiom — usually the identity axiom $1 \cdot \alpha = \alpha$ or distributivity. Check axiom 10 and axioms 7–8 first, as CSIR NET frequently targets these.
- A set with a "natural-looking" addition/multiplication is **not automatically** a vector space — every axiom must be verified, especially closure.
- Do not confuse a **vector space** with a **subspace** — every subspace is a vector space in its own right, but the reverse relationship (is W actually inside a larger V ?) is a separate question.
- $\{0\}$ alone (just the zero vector) is a valid vector space of dimension 0 — do not assume every vector space is non-trivial.

8. Quick Revision Summary

Concept	Key Fact
Structure	$(V, +)$ abelian group + compatible scalar action of field F
Total axioms	10 (5 for addition, 5 for scalar multiplication)
Zero vector	Unique; $0 \cdot \alpha = 0$ and $a \cdot 0 = 0$
Additive inverse	Unique; $(-1)\alpha = -\alpha$
No zero divisors	$a\alpha = 0 \implies a = 0$ or $\alpha = 0$
$\dim F^n$	n (over F itself)
$\dim M_{m \times n}(F)$	mn

$\dim P_n(F)$	$n + 1$
$\dim_{\mathbb{R}} \mathbb{C}$	2

CSIR NET

Linear Algebra

Vector Spaces

Axioms