

Subspace — CSIR NET Mathematical Sciences

Linear Algebra | Unit: Vector Spaces | Concept + Formulas + Solved Examples

1. Definition

DEFINITION

Let V be a vector space over a field F . A non-empty subset $W \subseteq V$ is called a **subspace** of V if W itself is a vector space over F under the same operations of addition and scalar multiplication inherited from V .

In practice, we never verify all 10 vector space axioms again — since $W \subseteq V$, most axioms (associativity, commutativity, distributivity, etc.) are automatically inherited. We only need to check **closure** and the presence of the **zero vector**.

2. Subspace Test (Most Important Tool)

THEOREM — SUBSPACE CRITERION

A non-empty subset W of a vector space $V(F)$ is a subspace of V if and only if:

1. $0 \in W$ (or $W \neq \emptyset$)
2. $\alpha, \beta \in W \implies \alpha + \beta \in W$ (closed under addition)
3. $\alpha \in W, c \in F \implies c\alpha \in W$ (closed under scalar multiplication)

ONE-STEP (COMBINED) TEST

W is a subspace of $V \iff W \neq \emptyset$ and for all $\alpha, \beta \in W, a, b \in F$:

$$a\alpha + b\beta \in W$$

This single condition is the fastest way to check subspaces in exam-solving.

3. Key Points to Remember

- Every vector space V has two **trivial subspaces**: $\{0\}$ (the zero subspace) and V itself.
- Any subspace other than $\{0\}$ and V is called a **proper (non-trivial) subspace**.
- The zero vector of W must be the **same** zero vector as in V .
- A subspace is always a vector space in its own right, hence its dimension satisfies $\dim W \leq \dim V$.
- The **intersection** of any collection of subspaces of V is again a subspace of V .
- The **union** of two subspaces is **generally NOT** a subspace (unless one contains the other).
- The **sum** $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is always a subspace, and in fact it is the *smallest* subspace containing $W_1 \cup W_2$.
- Solution set of a **homogeneous** system $Ax = 0$ is always a subspace of F^n ; solution set of a non-homogeneous system $Ax = b$ ($b \neq 0$) is **never** a subspace (fails to contain 0).
- A line or plane through the origin is a subspace of $\mathbb{R}^n$; a line/plane **not** through the origin is not.

4. Important Theorems & Formulas

4.1 Intersection of Subspaces

If W_1 and W_2 are subspaces of V , then $W_1 \cap W_2$ is also a subspace of V .

4.2 Union of Subspaces

$W_1 \cup W_2$ is a subspace of $V \iff W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

4.3 Sum of Subspaces

FORMULA

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$$

This is the smallest subspace of V containing both W_1 and W_2 .

4.4 Dimension Formula (Very Frequently Asked)

FORMULA

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

4.5 Direct Sum

DEFINITION

V is said to be the **direct sum** of subspaces W_1 and W_2 , written $V = W_1 \oplus W_2$, if:

$$V = W_1 + W_2 \quad \text{and} \quad W_1 \cap W_2 = \{0\}$$

FORMULA

$$\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$$

Equivalently: every $v \in V$ can be written **uniquely** as $v = w_1 + w_2$ with $w_1 \in W_1$, $w_2 \in W_2$.

4.6 Fundamental Subspaces of a Matrix (a favourite CSIR NET area)

Subspace	Definition	Dimension
Column space $\text{Col}(A)$	Span of columns of A	$\text{rank}(A)$
Row space $\text{Row}(A)$	Span of rows of A	$\text{rank}(A)$
Null space $N(A)$	$\{x : Ax = 0\}$	$n - \text{rank}(A)$ (Rank–Nullity)
Left null space	$\{y : y^T A = 0\}$	$m - \text{rank}(A)$

RANK–NULLITY THEOREM

For A an $m \times n$ matrix (or linear map $T : V \rightarrow W$, $\dim V = n$):

$$\text{rank}(A) + \text{nullity}(A) = n$$

5. Worked Examples

EXAMPLE 1 — BASIC SUBSPACE TEST

Check whether $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

Solution:

- $0 = (0, 0, 0) \in W$ since $0 + 0 + 0 = 0$. ✓
- Let $\alpha = (x_1, y_1, z_1), \beta = (x_2, y_2, z_2) \in W$. Then $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0 + 0 = 0$, so $\alpha + \beta \in W$. ✓
- For $c \in \mathbb{R}$: $cx_1 + cy_1 + cz_1 = c(x_1 + y_1 + z_1) = c \cdot 0 = 0$, so $c\alpha \in W$. ✓

Hence W is a subspace of $\mathbb{R}^3$ (in fact, it is a plane through the origin, $\dim W = 2$).

EXAMPLE 2 — A NON-EXAMPLE (COMMON TRAP)

Check whether $W = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$ is a subspace of $\mathbb{R}^2$.

Solution: Take $\alpha = (1, 1) \in W$ and $\beta = (1, -1) \in W$ (both satisfy $x^2 = y^2$). Then $\alpha + \beta = (2, 0)$, and $2^2 \neq 0^2$. So $\alpha + \beta \notin W$.

W is **not closed under addition** $\implies W$ is **not** a subspace, even though it contains 0 and is closed under scalar multiplication.

EXAMPLE 3 — UNION OF SUBSPACES IS NOT A SUBSPACE

Let $W_1 = \{(x, 0) : x \in \mathbb{R}\}$ (the x -axis) and $W_2 = \{(0, y) : y \in \mathbb{R}\}$ (the y -axis), both subspaces of $\mathbb{R}^2$.

$(1, 0) \in W_1 \subseteq W_1 \cup W_2$ and $(0, 1) \in W_2 \subseteq W_1 \cup W_2$, but their sum $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

Hence $W_1 \cup W_2$ is **not** a subspace — confirming the theorem in Section 4.2.

EXAMPLE 4 — DIMENSION FORMULA APPLICATION

In $\mathbb{R}^4$, let $W_1 = \text{span}\{(1, 0, 0, 0), (0, 1, 0, 0)\}$ and $W_2 = \text{span}\{(0, 1, 0, 0), (0, 0, 1, 0)\}$. Here $\dim W_1 = 2$, $\dim W_2 = 2$, and $W_1 \cap W_2 = \text{span}\{(0, 1, 0, 0)\}$ so $\dim(W_1 \cap W_2) = 1$.

By the formula:

$$\dim(W_1 + W_2) = 2 + 2 - 1 = 3$$

So $W_1 + W_2$ is a 3-dimensional subspace of $\mathbb{R}^4$ (not all of $\mathbb{R}^4$, and not a direct sum since $W_1 \cap W_2 \neq \{0\}$).

EXAMPLE 5 — SPACE OF MATRICES

Let $V = M_n(\mathbb{R})$, the vector space of all $n \times n$ real matrices. Show that the set of **symmetric matrices** $W = \{A \in V : A^T = A\}$ is a subspace.

Solution: The zero matrix is symmetric. If $A^T = A$ and $B^T = B$, then $(A + B)^T = A^T + B^T = A + B$, and $(cA)^T = cA^T = cA$. All three conditions hold, so W is a subspace, with

$$\dim W = \frac{n(n+1)}{2}$$

(Compare: skew-symmetric matrices form a subspace of dimension $\frac{n(n-1)}{2}$, and $M_n(\mathbb{R}) = \text{Sym} \oplus \text{Skew-Sym}$ is a classic direct sum decomposition.)

6. Common Pitfalls (Exam Traps)

WATCH OUT

- A set defined by an **inequality** (e.g. $x \geq 0$) is **never** a subspace — fails closure under scalar multiplication by negative scalars.
- A set defined by a **non-homogeneous** equation (e.g. $x + y = 1$) is **never** a subspace — does not contain 0.
- Sets involving **products/nonlinear conditions** (e.g. $xy = 0$, $x^2 + y^2 \leq 1$) generally fail closure under addition.
- $W_1 \cup W_2$ being a subspace is a **special case**, not the rule — always check containment first.
- Do not confuse "sum of subspaces" $W_1 + W_2$ with "direct sum" $W_1 \oplus W_2$ — the latter additionally needs $W_1 \cap W_2 = \{0\}$.

7. Quick Revision Summary

Concept	Key Formula / Condition
Subspace test	$0 \in W$, closed under $+$, closed under scalar mult.
Combined test	$a\alpha + b\beta \in W \quad \forall \alpha, \beta \in W, a, b \in F$
Intersection	Always a subspace
Union	Subspace $\iff$ one contains the other
Sum	$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$
Direct sum	$W_1 \cap W_2 = \{0\}$ and $\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$
Rank–Nullity	$\text{rank}(A) + \text{nullity}(A) = n$
Symmetric matrices	$\dim = \frac{n(n+1)}{2}$
Skew-symmetric matrices	$\dim = \frac{n(n-1)}{2}$

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