Unbiased Estimator

1. Definition

Let $X_1, X_2, ..., X_n$ be a random sample from a population depending on an unknown parameter θ. An estimator $T = T(X_1, ..., X_n)$ is called an **unbiased estimator** of θ if its expected value is equal to θ.

$$E_\theta[T] = \theta$$

Equivalently, the bias of T is zero:

$$\mathrm{Bias}(T) = E_\theta[T] - \theta = 0$$

2. Bias of an Estimator

For an estimator T of a parameter θ, the bias is defined by

$$\mathrm{Bias}(T) = E_\theta[T] - \theta$$

Therefore:

$$T \text{ is unbiased} \Leftrightarrow E_\theta[T] = \theta$$

3. Example: Sample Mean

Let $X_1, X_2, ..., X_n$ be a random sample with population mean μ. The sample mean is

$$X\blacksquare = (1/n)\, \Sigma_{i=1}^{n} X_i$$

Using linearity of expectation:

$$E[X\blacksquare] = (1/n)\, \Sigma\, E[X_i] = (1/n)(n\mu) = \mu$$

Hence,

$X\blacksquare$ is an unbiased estimator of μ.

4. Example: Sample Variance

Let the population variance be σ^2. The usual sample variance is

$$S^2 = 1/(n - 1)\, \Sigma_{i=1}^{n}(X_i - X\blacksquare)^2$$

For a random sample with finite variance σ^2,

$$E[S^2] = \sigma^2$$

Therefore, S^2 is an unbiased estimator of σ^2.

Important: If we instead use the maximum-likelihood form

$$V = 1/n\, \Sigma_{i=1}^{n}(X_i - X\blacksquare)^2$$

then $E[V] = ((n - 1)/n)\sigma^2$, so V is biased downward for σ^2.

5. Example: Estimating a Function of a Parameter

If T is an unbiased estimator of θ, it does not generally follow that $g(T)$ is an unbiased estimator of $g(\theta)$.

For example, if $X\blacksquare$ is an unbiased estimator of μ, then generally

$$E[(X\blacksquare)^2] \neq \mu^2$$

In fact,

$$E[(X■)^2] = Var(X■) + \mu^2$$

Thus, $X■^2$ is generally a biased estimator of μ^2.

6. How to Check Whether an Estimator Is Unbiased

Step	What to do
1	Write down the proposed estimator T.
2	Find E[T].
3	Compare E[T] with the parameter being estimated.
4	If $E[T] = \theta$, the estimator is unbiased; otherwise it is biased.

7. Worked Example

Suppose $X_1, ..., X_n$ have mean μ. Consider the estimator

$$T = (2X■ + 1)/2$$

Since $E[X■] = \mu$,

$$E[T] = (2E[X■] + 1)/2 = \mu + 1/2$$

Since $E[T] \neq \mu$, T is a biased estimator of μ. Its bias is

$$Bias(T) = 1/2$$

8. Important Properties for CSIR-NET

• The expectation of an unbiased estimator equals the parameter it estimates.
• An estimator may be unbiased even though it is not the only unbiased estimator.
• A biased estimator can sometimes have smaller variance than an unbiased estimator.
• Unbiasedness concerns the expectation of the estimator, not the value obtained from one sample.
• If T is unbiased for θ and a, b are constants, then $aT + b$ is unbiased for $a\theta + b$.

9. Linear Transformation Property

If $E[T] = \theta$, then for constants a and b,

$$E[aT + b] = aE[T] + b = a\theta + b$$

Hence, $aT + b$ is an unbiased estimator of $a\theta + b$.

10. Quick Revision

Definition:

$$T \text{ is unbiased for } \theta \Leftrightarrow E[T] = \theta$$

Bias:

$$Bias(T) = E[T] – \theta$$

Unbiasedness:

$$\text{Bias}(T) = 0$$

Key examples:

X■ is unbiased for μ.
$S^2 = 1/(n-1)\, \Sigma(X_i - X$■$)^2$ is unbiased for σ^2.
$1/n\, \Sigma(X_i - X$■$)^2$ is biased for σ^2.

CSIR-NET Shortcut

Whenever a question asks whether an estimator is unbiased, calculate its expectation first.