Eisenstein's Criterion for Irreducible Polynomials

1. Eisenstein's Criterion

Let $\mathbf{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$ be a polynomial in Z[x]. If there exists a prime number p such that the following conditions hold, then f(x) is irreducible over Q:

$$p \mid a_0,\ p \mid a_1,\ \ldots,\ p \mid a_{n-1}$$

$$p \text{ does not divide } a_n$$

$$p^2 \text{ does not divide } a_0$$

Therefore: f(x) is irreducible over Q.

2. Conditions to Remember

Condition	Requirement
1	p divides every coefficient except the leading coefficient
2	p does not divide the leading coefficient
3	p^2 does not divide the constant term

3. Example

Consider $\mathbf{f(x) = x■ + 6x^3 + 12x^2 + 18x + 6}$. Choose the prime $\mathbf{p = 2}$.

2 divides 6, 12, 18, and 6.

2 does not divide 1.

$2^2 = 4$ does not divide 6.

All Eisenstein conditions are satisfied. Hence, $\mathbf{x■ + 6x^3 + 12x^2 + 18x + 6}$ **is irreducible over Q.**

4. Important Point

Failure of Eisenstein's Criterion does **NOT** imply that the polynomial is reducible.

$$\text{Eisenstein fails} \Rightarrow \text{reducible}$$

The above implication is false. Failure only means that Eisenstein's Criterion cannot be used directly.

5. Example Where Eisenstein Fails

Consider $\mathbf{f(x) = x^2 + x + 2}$. Eisenstein's Criterion cannot be applied directly.

$$\Delta = b^2 – 4ac = 1^2 – 4(1)(2) = –7$$

Since –7 is not a square in Q, the quadratic has no rational roots. Therefore, $\mathbf{x^2 + x + 2}$ **is irreducible over Q.**

6. Shifted Eisenstein Criterion

Sometimes Eisenstein's Criterion does not apply directly to f(x), but it applies to a shifted polynomial f(x + a). If f(x + a) is irreducible over Q, then f(x) is also irreducible over Q.

Useful substitution: $x \to x + a$

7. Reducible and Irreducible Polynomials

A polynomial $f(x) \in F[x]$ is reducible over F if it can be written as $f(x) = g(x)h(x)$, where $g(x), h(x) \in F[x]$ and both have positive degree.

$$x^2 - 4 = (x - 2)(x + 2)$$

Therefore, $x^2 - 4$ is reducible over Q.

$$x^2 + 1$$

This polynomial has no rational root, so it is irreducible over Q.

8. CSIR-NET Shortcut

For $f(x) = a_n x^n + \dots + a_1 x + a_0$, look for a prime p satisfying:

$$p \text{ divides } a_0, a_1, \dots, a_{n-1}$$

$$p \text{ does not divide } a_n$$

$$p^2 \text{ does not divide } a_0$$

Then immediately conclude: f(x) is irreducible over Q.

Memory Trick

All non-leading coefficients $\to$ divisible by p
Leading coefficient $\to$ not divisible by p
Constant term $\to$ not divisible by p^2